Multisoliton complexes moving on a cnoidal wave background

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We obtain solutions of coupled nonlinear Schrödinger equations which describe multisoliton complexes moving on a cnoidal-wave background. Our method is based on the Darboux transformation, which uses Sym's solution of the associated linear equation. Solutions are presented in a matrix determinant form, matrix elements of which are expressed in terms of Jacobi's elliptic functions. Some characteristics of multisoliton complexes like widths and amplitudes are explicitly calculated.

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I. INTRODUCTION

Recently there has arisen much interest in multisoliton complexes (MSC's) [1,2], which appear in various physical problems. These include solitons in multicore fiber devices [3], multicomponent Bose-Einstein condensates in a trapped ultracold gas [4], gap solitons [5], and incoherent solitons in photorefractive materials [2,6]. An MSC is a self-localized state, which is a nonlinear superposition of several single solitons of the system having the same speed and thus moves as a single complex. Generally, MSC's are described by coupled nonlinear Schrödinger equations (NLSE's). A number of publications dealt with the explicit construction of MSC's of coupled NLSE's. They include soliton solutions [7,8] and periodic solutions [9–11] of the two-component case and solutions of equations having components larger than 2 [9,12,2].

In some special cases, like the wave propagations in a homogeneous medium having a Kerr-type nonlinear response, the corresponding NLSE's reduce to the integrable equation

$$\partial_x \psi_k = i\partial_z^2 \psi_k + 2i\sigma \sum_{i=0}^{N} |\psi_i|^2 \psi_k, \quad k=0,\ N \text{ and } \sigma = \pm 1. \tag{1}$$

The integrability allows us to use the inverse scattering method to construct solitons [13]. But the high-level mathematical technicality of the method makes it difficult for finding more complex solutions, multisolitons and/or solitons having nonvanishing backgrounds. Thus, most solutions of MSC's have been constructed in a form of stationary solutions or using the linear superposition principle [14]. The stationary MSC reduces the problem of the coupled NLSE's to a set of ordinary differential equations. Some important results obtained in these ways are solitary-wave solutions [15], MSC solutions of partially coherent solitons in [16], MSC's on a background [17,18], and MSC's in a sea of radiation modes [19]. Collisions of MSC's are also investigated and illustrated by numerical examples [20].

Especially, the MSC solutions on a background [17,18] provide answers on how MSC's interact with plane waves and/or the change of characteristics of MSC's by an addition of plane waves. These solutions can be useful, for example, in the theory of dark incoherent solitons. At this point, we note that the plane-wave solution of the NLSE's is a special limit of more general solutions having nonzero asymptote, the so-called *cnoidal-wave* solutions of integrable theories. Then, it would be very interesting to know the characteristics of MSC's lying on a cnoidal wave of pulsating amplitude, instead of a plane wave of constant amplitude. A related experiment is the appearance of a cnoidal-wave train in the Maxwell-Bloch equation [21]. More close experimental development is the formation of solitons in nonlinear periodic structures, known as gap solitons [22].

In fact, there have been various special periodic solutions of multicomponent NLSE's (a cnoidal wave is a specific case), which were obtained by various direct methods using the appropriate ansatz [9,23]. For example, the Hirota bilinear method was used to find periodic solutions of coupled NLSE's [24,25]. In Ref. [2], Porubov and Parker employ the following ansatz:

$$\psi_0 = \sqrt{F\wp^2 + A\wp + B}\, e^{i\theta(z,x)}, \quad \psi_1 = \sqrt{G\wp^2 + D\wp + E}\, e^{i\phi(z,x)}, \tag{2}$$

where $\wp = \wp(x-cz, g_2, g_3)$ is the Weierstrass function. But all components of these solutions (both $|\psi_0|, |\psi_1|$) are periodic over the entire x axis and are not forms of the MSC's (a soliton for ψ_1 in this case) plus a periodic wave (ψ_0). These types of solutions are not suitable to the study of the effects of periodic waves applied to an MSC.

In this paper, we will calculate MSC solutions on a cnoidal-wave background using a method based on the Darboux transformation (DT), instead of using some special ansatz. The DT gives a way to obtain a new solution of "a soliton+starting solution" type when it is applied once on a given starting solution [26–28]. In fact, this method is applied to the vector NLSE (corresponding to the $N=1$ case of the present paper) to study a soliton coupled to a cnoidal-wave background [29,30]. To create an MSC on a starting solution $\psi^{(0)}$ of the cnoidal wave [we are restricted by the fact that a starting solution has a ψ_0 component only in Eq. (1)], we apply an N iteration of the DT on $\psi^{(0)}$. This proce-

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dure is described in [31], which shows that the final result of "MSC+starting solution" can be given in a closed determinant form.

Section II explains the DT method for constructing the MSC on a background, expressed in a closed determinant form. Section III shows $N=1,2,3$ MSC's on a dn-type cnoidal wave in a focusing medium ($\sigma=1$). Some characteristics of MSC's like the width and amplitude variation along with the modulus parameter k of dn wave backgrounds are discussed. The shift of crests of the dn wave is also calculated. MSC's lying on the plane wave (corresponding to the $k=0$ limit) and a soliton ($k=1$ limit) of the dn cnoidal wave are described. Section IV discusses MSC's of cn background in a focusing medium ($\sigma=1$). It shows that there are two possible type of MSC's on a cn-type background. Section V discusses MSC's of sn background in a defocusing medium ($\sigma=-1$). Section VI contains discussions. The derivation of Sym's solution is shown in Appendix A and the calculation of the shifts of crests is given in Appendix B. All the results and figures of the paper are checked and prepared using the symbolic package MATHEMATICA.

II. MULTISOLITON COMPLEXES IN A MATRIX DETERMINANT FORM

Recently, a simple but powerful MSC finding method was developed in [31]. The method, which is based on the Darboux transformation, uses Crum's formula and avoids the stationary ansatz [26–28]. More explicit closed expressions of the MSC's having complex behavior were found using this method. This method is suited well to the construction of N-vector MSC's on an arbitrary nonvanishing background, especially on the cnoidal-wave background. Here, we briefly explain the method of finding MSC solutions using the Darboux transformation. A more detailed exposition can be found in [31]. We first write down the associated linear equation (Lax pair) of the coupled NLSE (1) in a form needed in the present paper:

$$\partial_z a - \sigma\psi b + \beta a/2 = 0, \quad \partial_z b + \psi^* a - \beta b/2 = 0,$$

$$\partial_x a - i\sigma|\psi|^2 a - i\sigma(\partial_z\psi)b + i\sigma\beta\psi b - i\beta^2 a/2 = 0,$$

$$\partial_x b + i\sigma|\psi|^2 b - i(\partial_z\psi^*)a - i\beta\psi^* a + i\beta^2 b/2 = 0,$$

$$\partial_z c - \beta c/2 = 0, \quad \partial_x c + i\beta^2 c/2 = 0, \tag{3}$$

where β is an arbitrary parameter and a,b,c are solutions of the linear equations corresponding to the parameter β. The signature σ is either 1 or -1 depending on whether the group velocity dispersion is abnormal ($\sigma=1$) or normal ($\sigma=-1$) or the waveguide is self-focusing ($\sigma=1$) or self-defocusing ($\sigma=-1$). Then the DT procedure to obtain MSC solutions is given in the following steps.

First, choose a particular solution for $\psi=\psi^{(0)}$, which describe a cnoidal-wave background. The cnoidal-wave background $\psi^{(0)}$ will be given in terms of Jacobi's elliptic functions like dn, sn, and cn.

Next, we integrate the linear equation (3) for the given background $\psi^{(0)}$ to obtain N solutions $a=a_j$, $b=b_j$, and $c=c_j$ for N values of $\beta=\beta_j$, $j=1,N$. Here we take $\beta_j=$real $j=1,N$ to get stationary MSC's.

Finally, the Darboux transformation gives a MSC solution in terms of $a_j,b_j,c_j,j=1,N$ by the following matrix determinant form. See more details in [31]. We first define two quantities D and Q from which the MC solutions can be constructed. Let D for $N=1,2,3$ be defined as

$$D^{(N=1)} = LP_1,$$

$$D^{(N=2)} = L\begin{vmatrix} -\beta_1 M_1 & -\beta_2 P_2 \\ P_1 & M_2 \end{vmatrix} + \sigma R\left\|\begin{matrix} \alpha_1 & \alpha_2 \\ \kappa_1 & \kappa_2 \end{matrix}\right\|^2,$$

$$\begin{aligned} D^{(N=3)} = {} & L\begin{vmatrix} (-\beta_1)^2 P_1 & (-\beta_2)^2 M_2 & (-\beta_3)^2 P_3 \\ -\beta_1 M_1 & -\beta_2 P_2 & -\beta_3 M_3 \\ P_1 & M_2 & P_3 \end{vmatrix} \\ & + \sigma R(\beta_1^2-\beta_3^2)(\beta_2^2-\beta_3^2)\left\|\begin{matrix} \alpha_1 & \alpha_2 \\ \kappa_1 & \kappa_2 \end{matrix}\right\|^2 \frac{P_3}{\beta_3} + \sigma R(\beta_2^2-\beta_1^2) \\ & \times(\beta_3^2-\beta_1^2)\left\|\begin{matrix} \alpha_2 & \alpha_3 \\ \kappa_2 & \kappa_3 \end{matrix}\right\|^2 \frac{P_1}{\beta_1} + \sigma R(\beta_1^2-\beta_2^2)(\beta_3^2-\beta_2^2) \\ & \times\left\|\begin{matrix} \alpha_1 & \alpha_3 \\ \kappa_1 & \kappa_3 \end{matrix}\right\|^2 \frac{M_2}{\beta_2}, \end{aligned} \tag{4}$$

where

$$L = i^{N^2(N-1)/2}\prod_{i=1,N}\alpha_i^* \prod_{j=1,N,\ j>i}(\beta_j-\beta_i)^{N+1}(\beta_j+\beta_i)^3,$$

$$R = -4L\frac{\prod\beta_i}{\prod_{j<i}(\beta_j-\beta_i)}, \tag{5}$$

and $P_i=|\alpha_i|^2+\sigma|\kappa_i|^2+\sigma|\zeta_i|^2$, $M_i=|\alpha_i|^2+\sigma|\kappa_i|^2-\sigma|\zeta_i|^2$, $\alpha_i=a_i/l_i$, $\kappa_i=b_i/l_i$, $\zeta_i=c_i/m_i$,

$$l_i = \sqrt{\prod_{j=1,N\ j\neq i}(\beta_i+\beta_j)}, \quad m_i = \sqrt{\prod_{j=1,N\ j\neq i}|\beta_i-\beta_j|}. \tag{6}$$

Here $\|\cdots\|^2$ means the squared absolute of a determinant. Another quantity Q is defined by the form

$$Q_0^{(N=1)} = 2i\sigma L\beta_1 a_1 b_1^*, \quad |Q_1^{(N=1)}| = 2i\sigma L\beta_1 a_1 c_1^*,$$

$$Q_0^{(N=2)} = -2i\sigma L\begin{vmatrix} \beta_1 a_1 b_1^* & \beta_2 a_2 b_2^* \\ P_1 & M_2 \end{vmatrix},$$

$$\begin{aligned} Q_i^{(N=2)} = {} & -2i\sigma L\begin{vmatrix} \delta_{i,1}\beta_1 a_1 c_1^* & \delta_{i,2}\beta_2 a_2 c_2^* \\ P_1 & M_2 \end{vmatrix} + iRl_i\begin{vmatrix} \alpha_1 & \alpha_2 \\ \kappa_1 & \kappa_2 \end{vmatrix} \\ & \times\begin{vmatrix} \delta_{i,1}c_1^* & \delta_{i,2}c_2^* \\ \kappa_1^* & \kappa_2^* \end{vmatrix}, \quad i=1,2, \end{aligned}$$

$$Q_0^{(N=3)} = 2i\sigma L \begin{vmatrix} \beta_1 a_1 b_1^* & \beta_2 a_2 b_2^* & \beta_3 a_3 b_3^* \\ -\beta_1 M_1 & -\beta_2 P_2 & -\beta_3 M_3 \\ P_1 & M_2 & P_3 \end{vmatrix} + 2iR \begin{vmatrix} \beta_1^2\alpha_1 & \beta_2^2\alpha_2 & \beta_3^2\alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix} \begin{vmatrix} \beta_1^2\kappa_1^* & \beta_2^2\kappa_2^* & \beta_3^2\kappa_3^* \\ \alpha_1^* & \alpha_2^* & \alpha_3^* \\ \kappa_1^* & \kappa_2^* & \kappa_3^* \end{vmatrix},$$

$$Q_i^{(N=3)} = 2i\sigma L \begin{vmatrix} \delta_{i,1}\beta_1 a_1 c_1^* & \delta_{i,2}\beta_2 a_2 c_2^* & \delta_{i,3}\beta_3 a_3 c_3^* \\ -\beta_1 M_1 & -\beta_2 P_2 & -\beta_3 M_3 \\ P_1 & M_2 & P_3 \end{vmatrix} + iRl_i \frac{(\beta_1^2-\beta_3^2)(\beta_2^2-\beta_3^2)}{(\beta_i+\beta_3)\beta_3} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \kappa_1 & \kappa_2 \end{vmatrix} \times \begin{vmatrix} \delta_{i,1}c_1^* & \delta_{i,2}c_2^* \\ \kappa_1^* & \kappa_2^* \end{vmatrix} P_3 + iRl_i \frac{(\beta_2^2-\beta_1^2)(\beta_3^2-\beta_1^2)}{(\beta_i+\beta_1)\beta_1} \begin{vmatrix} \alpha_2 & \alpha_3 \\ \kappa_2 & \kappa_3 \end{vmatrix} \times \begin{vmatrix} \delta_{i,2}c_2^* & \delta_{i,3}c_3^* \\ \kappa_2^* & \kappa_3^* \end{vmatrix} P_1 + iRl_i \frac{(\beta_1^2-\beta_2^2)(\beta_3^2-\beta_2^2)}{(\beta_i+\beta_2)\beta_2} \begin{vmatrix} \alpha_1 & \alpha_3 \\ \kappa_1 & \kappa_3 \end{vmatrix} \times \begin{vmatrix} \delta_{i,1}c_1^* & \delta_{i,3}c_3^* \\ \kappa_1^* & \kappa_3^* \end{vmatrix} M_2 + 2iRl_i \begin{vmatrix} \beta_1^2\alpha_1 & \beta_2^2\alpha_2 & \beta_3^2\alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix} \times \begin{vmatrix} \delta_{i,1}c_1^* & \delta_{i,2}c_2^* & \delta_{i,3}c_3^* \\ \alpha_1^* & \alpha_2^* & \alpha_3^* \\ \kappa_1^* & \kappa_2^* & \kappa_3^* \end{vmatrix}, \quad i=1,2,3. \tag{7}$$

Using these quantities, we can obtain MSC solutions using

$$\psi_0 = \psi^{(0)} + i\sigma\frac{Q_0}{D}, \quad \psi_i = i\sigma\frac{Q_i}{D}, \quad i=1,N. \tag{8}$$

III. MULTISOLITON COMPLEXES ON A dn-TYPE BACKGROUND

Soliton complexes on a dn-type background can be obtained by choosing a starting solution

$$\psi^{(0)} = \exp\{ip^2(2-k^2)x\}p\ \mathrm{dn}(pz,k), \tag{9}$$

where $\mathrm{dn}=\mathrm{dn}(pz,k)$, cn, and sn are the standard Jacobi elliptic functions with the modulus k and p is an arbitrary constant. In the following, we employ the terminology and notation of Ref. [32] as far as elliptic functions are involved. This is the well-known cnoidal solution of the nonlinear Schrödinger equation in a focusing medium ($\sigma=1$). The linear equation (3) is with a dn-type background $\psi=\psi^{(0)}$ in Eq. (9) was first integrated by Sym in a different context (description of vortex motion in hydrodynamics) [33]. It was then applied to an NLSE-related problem in Refs. [34,29,30].

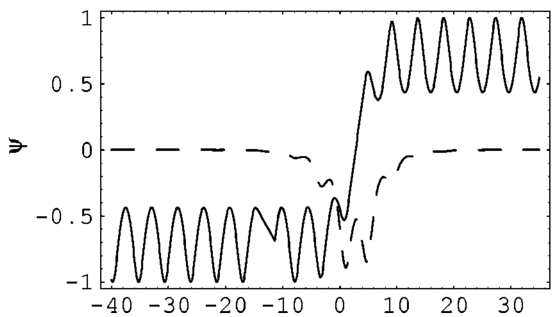

FIG. 1. A bright soliton on a dn background. The solid line is for ψ_0 and the dashed line is for ψ_1 at $x=0$ with parameters $p=1$, $k=0.9$, $r_1=1$, $u_1=2.9$, and $\beta_1=1.87$.

The detailed proof of Sym's solution is given in Appendix A. Explicitly, it is

$$a = \exp\{ip^2(2-k^2)x/2\}\exp(i\gamma x + \delta pz)\frac{\Theta(pz-u)}{\Theta(pz)},$$

$$b = -\exp\{-ip^2(2-k^2)x\}\frac{\mathrm{sn}(u)\mathrm{dn}(pz-u)}{\mathrm{cn}(u)}a,$$

$$c = r\exp(\beta z/2 - i\beta^2 x/2), \tag{10}$$

where u,r are arbitrary constants. The Bäcklund parameter β is related to the real parameter u as

$$\beta = -p\frac{\mathrm{dn}(u)}{\mathrm{sn}(u)\mathrm{cn}(u)}, \tag{11}$$

and γ,δ in Eq. (10) are related to u as

$$\gamma = -\frac{p^2}{2}\left(\frac{\mathrm{dn}^2(u)}{\mathrm{cn}^2(u)} - \frac{1}{\mathrm{sn}^2(u)}\right),$$

$$\delta = \frac{\Theta'(u)}{\Theta(u)} + \frac{1}{2}\frac{\mathrm{dn}(u)}{\mathrm{sn}(u)\mathrm{cn}(u)} - \frac{\mathrm{sn}(u)\mathrm{dn}(u)}{\mathrm{cn}(u)}, \tag{12}$$

where the Jacobi theta function is defined by the complete elliptic integral of the first (second) kind K (E) as

$$\Theta(u) = \theta_4\left(\frac{\pi u}{2K}\right) = 1 + 2\sum(-1)^n q^{n^2}\cos\left(\frac{n\pi u}{K}\right), \tag{13}$$

with $q=\exp(-\pi K'/K)$ and $K' \equiv K(k'=\sqrt{1-k^2})$.

Figure 1 shows the simplest $N=1$ case of an MSC on a dn-type background at $x=0$, which was drawn using

$$\psi_0 = \exp[ip^2(2-k^2)x]p\ \mathrm{dn}(pz,k) + iQ_0^{(N=1)}/D^{(N=1)},$$

$$\psi_1 = iQ_1^{(N=1)}/D^{(N=1)}. \tag{14}$$

It is constituted of a dark soliton on a cnoidal background (solid line) plus a bright soliton (dashed line). To obtain this figure, we use the symbolic package MATHEMATICA, which is also used to check the correctness of the obtained solution. In Fig. 1, we can see that ψ_0 becomes a cnoidal wave when we move away from the soliton. Appendix B shows that $\psi_0 \to \psi^{(0)} = p\exp\{ip^2(2-k^2)x\}\mathrm{dn}(pz)$ for $z\to\infty$ and ψ_0

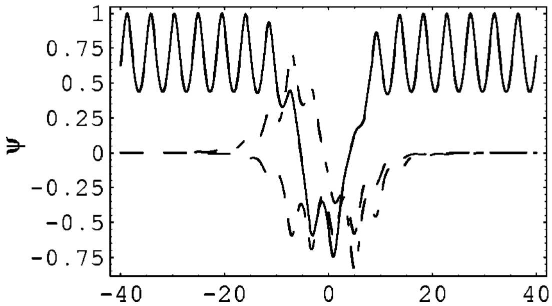

FIG. 2. Two bright solitons on a dn-type background. The solid line is for ψ_0, the dashed line is for ψ_1, and the dot-dashed line is for ψ_2 at $x=0$ with parameters $p=1$, $k=0.9$, $u_1=2.9$, $u_2=2.8$, $\beta_1=1.87$, $\beta_2=2.14$, $r_1=r_2=1$.

$\rightarrow -p\exp\{ip^2(2-k^2)x\}\mathrm{dn}(pz-2u)$ for $z\rightarrow-\infty$. This calculation shows that crests of the cnoidal wave are shifted by $2u/p$ across the soliton.

Figure 2 shows "a dn-type dark soliton+two bright solitons" from ψ_0,ψ_1,ψ_2 of Eq. (8) which corresponds to a $N=2$ MSC on a dn background. To obtain this figure, we need two real parameters $u,i=1,2$. They give two sets of a_i,b_i,c_i $i=1,2$ by substituting $u\rightarrow u_i$ $i=1,2$, in Eq. (10). Substituting these values into the formula of Eqs. (4) and (7) with ($N=2$), we can get MSC's constituted of "a dark+two bright" oscillating solitons.

As was discussed in the $N=1$ case, ψ_0 in Fig. 2 ($N=2$) becomes a cnoidal wave when $z\rightarrow\pm\infty$, though there is a relative shift of crests. In this case, each soliton in the $N=2$ MSC contributes $2u_i$ to the shift of crests, such that $\psi_0\rightarrow(-1)^2p\exp\{ip^2(2-k^2)x\}\mathrm{dn}(pz-2u_1-2u_2)$ for $z\rightarrow-\infty$ and $\psi_0\rightarrow p\exp\{ip^2(2-k^2)x\}\mathrm{dn}(pz)$ for $z\rightarrow\infty$. Generally the nonlinearly superposed solitons do not interfere with each other in the region $z\rightarrow-\infty$, and each contributes independently to the shift of crests of the cnoidal wave ψ_0 such that $\psi_0\rightarrow(-1)^Np\exp\{ip^2(2-k^2)x\}\mathrm{dn}(pz-2\Sigma_{i=1}^N u_i)$ for $z\rightarrow-\infty$. This fact will be seen again in following figures. We confirm this fact explicitly using the symbolic package MATHEMATICA by applying it to Eq. (14).

We can obtain MSC's lying on a plane-wave background by taking the $k\rightarrow 0$ limit of the previous results. In this limit, $\psi^{(0)}\rightarrow p\exp(2ip^2x)$, $q\rightarrow 0$, $\Theta(u)\rightarrow 1$, and

$$a=\exp\{ip^2[2+\csc^2(u)-\sec^2(u)]x/2+p[\csc(u)\sec(u)/2-\tan(u)]z\},$$

$$b=-\tan(u)\exp\{ip^2[-2+\csc^2(u)-\sec^2(u)]x/2+p[\csc(u)\sec(u)/2-\tan(u)]z\},$$

$$c=r\exp[-ip^2\csc^2(u)\sec^2(u)x/2-p\csc(u)\sec(u)z/2],$$

$$\beta=-p\csc(u)\sec(u). \quad (15)$$

To get a simple type solution, we take

$$\cot u_2=2\cot u_1,\quad r_1^2=-\sec^2 u_1/3,\quad r_2^2=-(3+\sec^2 u_1)/12. \quad (16)$$

Substituting all these results into Eq. (8), we obtain

$$\psi_0=-\frac{1}{2}p\exp(2ip^2x)\{1-3\tanh^2[p\cot(u_1)z]\},$$

$$\psi_1=\sqrt{3}p\csc(u_1)\exp\{ip^2[1+\csc^2(u_1)]x\}\times\mathrm{sech}[p\cot(u_1)z]\tanh[p\cot(u_1)z],$$

$$\psi_2=\frac{\sqrt{3}}{2}p\cot(u_1)\sqrt{3+\sec^2(u_1)}\exp\{2ip^2[2\csc^2(u_1)-1]x\}\times\mathrm{sech}^2(p\cot u_1\, z). \quad (17)$$

At this point, we note that cn- and sn-type backgrounds, which will be dealt with in the following sections, do not have the plane-wave background limit.

By taking the $k\rightarrow 1$ limit on the $N=1$ formula, we can get a MSC where two solitons interact coherently through the ψ_0 component. In this limit, $\psi^{(0)}\rightarrow p\,\mathrm{sech}(pz)\exp(ip^2x)$, $q\rightarrow 1$, $K\rightarrow\infty$, $K'\rightarrow\pi/2$. And (here, we express $u_1\rightarrow u, a_1\rightarrow a, r_1\rightarrow r$ for simplicity)

$$\Theta(u)\rightarrow 2\sqrt{K/K'}\exp\left(-\frac{\pi K}{4K'}\right)\cosh(u) \quad (18)$$

and

$$a=\exp[ip^2\coth^2(u)x/2+p\coth(u)z/2]\cosh(pz-u)\mathrm{sech}(pz),$$

$$b=-\exp\{ip^2[\coth^2(u)-2]x/2+p\coth(u)z/2\}\sinh(u)\mathrm{sech}(pz),$$

$$c=r\exp[-ip^2\coth^2(u)x/2-p\coth(u)z/2],$$

$$\beta=-p\coth(u). \quad (19)$$

Substituting all these results into Eq. (8), we obtain

$$\psi_0=p\exp(ip^2x)\times\frac{r^2-\exp[2p\coth(u)z]}{\exp[2p\coth(u)z]\cosh(pz-2u)+r^2\cosh(pz)},$$

$$\psi_1=2pr\coth u\frac{\exp[ip^2\coth^2(u)x+p\coth(u)z]}{\exp[2p\coth(u)z]\cosh(pz-2u)+r^2\cosh(pz)}. \quad (20)$$

Similarly, taking three parameters u_i, $i=1,3$, and using Eq. (10), we can obtain a MSC of "a dark+three bright oscillating solitons"; see Fig. 3. The shape of ψ_3 in Fig. 3 for $z\rightarrow\infty$ becomes a sech type, as will be explained in the following. For parameters of Fig. 3, $c_3(a_3)(b_3)$ term dominates $c_1,c_2(a_1,a_2)(b_1,b_2)$ terms for $z\rightarrow\infty$. Using this fact and Eqs. (8), (10), and (11), we can obtain

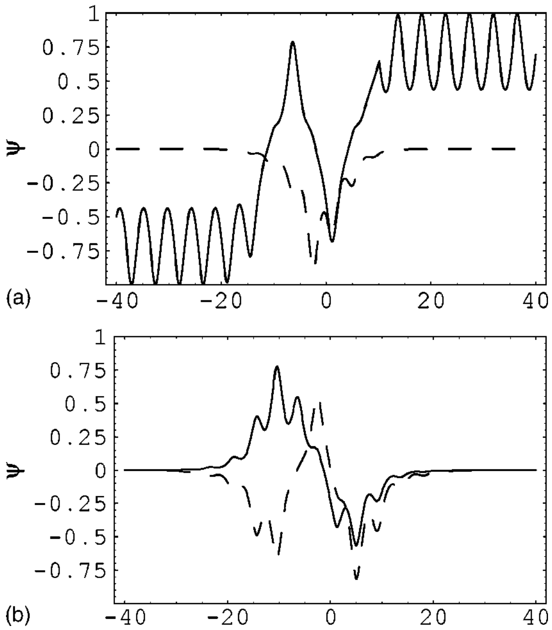

FIG. 3. Four-component soliton MSC constructed by adding three solitons on a dn background. (a) Solid line: ψ_0. Dashed line: ψ_1. (b) Solid line: ψ_2. Dashed line: ψ_3. The parameters are $p=1$, $k=0.9$, $u_i=3.1, 2.9, 2.8$, $\beta_i=1.58, 1.87, 2.14$, $r_i=0$, $i=1,3$.

$$\psi_3 \to -\frac{2\beta_3 a_3 c_3^*}{|\alpha_3|^2+|\kappa_3|^2+|\zeta_3|^2} = A\exp(-i\theta)\mathrm{sech}(wz+\eta) \tag{21}$$

(here, $\theta=0$; $\theta\neq 0$ is used in the following sections),

$$A = p\frac{\mathrm{dn}(u_3)}{\mathrm{sn}(u_3)\mathrm{cn}(u_3)} \Bigg/ \sqrt{1+\frac{\mathrm{sn}^2(u_3)\mathrm{dn}^2(pz-u_3)}{\mathrm{cn}^2(u_3)}},$$

$$w = p\left(\frac{\Theta'(u_3)}{\Theta(u_3)} + \frac{\mathrm{cn}(u_3)\mathrm{dn}(u_3)}{\mathrm{sn}(u_3)}\right), \tag{22}$$

$$\exp\,\eta = \frac{\Theta(pz-u_3)}{r_3\Theta(pz)}\frac{(\beta_3-\beta_1)(\beta_3-\beta_2)}{(\beta_3+\beta_1)(\beta_3+\beta_2)} \times \sqrt{1+\frac{\mathrm{sn}^2(u_3)\mathrm{dn}^2(pz-u_3)}{\mathrm{cn}^2(u_3)}}. \tag{23}$$

The amplitude A oscillates along z with periodicity $2K/p$, with a maximum value of $p/\mathrm{sn}(u_3)$ and a minimum value of $p\mathrm{dn}(u_3)/\mathrm{sn}(u_3)$. We thus define $\overline{A}\equiv[1+\mathrm{dn}(u_3)]/[2\,\mathrm{sn}(u_3)]$. Especially at $k\to 0$, $\overline{A}\to p/\sin(u_3)$. The width w is a monotonically decreasing function of u in $0\leqslant u\leqslant 2K$, with $w=\infty$ at $u=0$, $w=0$ at $u=K$, and $w=-\infty$ at $u=2K$. Especially at $k\to 0$, $w\to p\cot u_3$. In Fig. 4, we can see two plots, $\overline{A}$ vs w for a given k [Fig. 4(a); the solid line is for $k=0.5$, and the dotted line is for $k=0.9$] and $\overline{A}$ vs k for a given w [Fig. 4(b); the solid line is for $w=1$, and the dotted line is for $w=1.3$]. These figures show that the cnoidal wave ($k\neq 0$) makes solitons broader in width and smaller in amplitude, compared to the plane wave ($k=0$). These effects are more apparent for larger k.

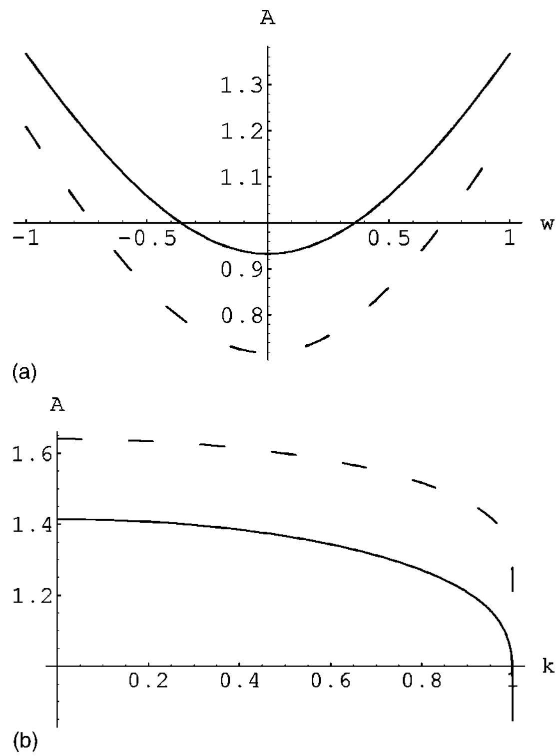

FIG. 4. Amplitude of the soliton on a dn background: (a) amplitude vs width w; the solid line is for $k=0.5$, and the dashed line is for $k=0.9$ ($p=1$). (b) Amplitude vs k; the solid line is for $w=1$ and the dashed line is for $w=1.3$.

IV. MULTISOLITON COMPLEXES ON A cn-TYPE BACKGROUND

Soliton complexes on a cn-type background can be obtained by choosing a starting solution

$$\psi^{(0)} = \exp\{ip^2(2k^2-1)x\}kp\,\mathrm{cn}(pz,k). \tag{24}$$

This is another cnoidal solution of the NLSE of the focusing medium.

A. Type I

The linear Equation (3) is integrated with the result

$$a = \exp\{ip^2(2k^2-1)x/2\}\exp(i\gamma x+\delta pz)\frac{\Theta_c(pz-u)}{\Theta_c(pz)},$$

$$b = -\exp\{-ip^2(2k^2-1)x\}k\frac{\mathrm{sn}(u)\mathrm{cn}(pz-u)}{\mathrm{dn}(u)}a,$$

$$c = r\exp(\beta z/2 - i\beta^2 x/2). \tag{25}$$

The Bäcklund parameter β is in this case given by

$$\beta = -p\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)\mathrm{dn}(u)}, \tag{26}$$

and γ, δ in Eq. (25) are

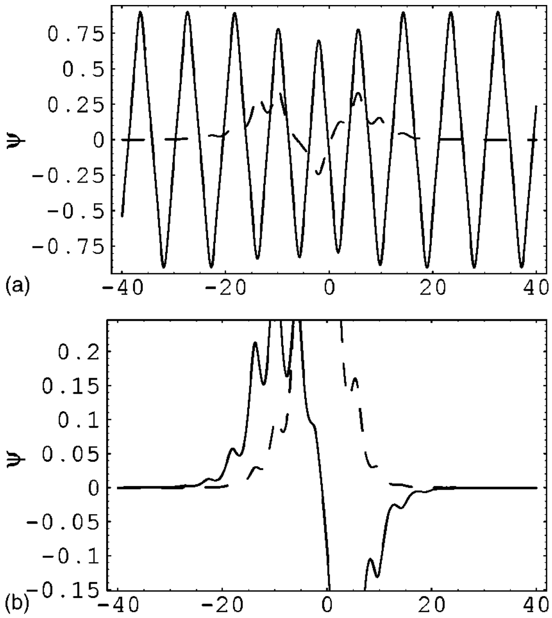

FIG. 5. Four-component MSC constructed by adding three solitons on a cn background (type I). (a) Solid line: ψ_0. Dashed line: $\psi_1 \exp(0.78i)$. (b) Solid line: $\psi_2 \exp(0.83i)$. Dashed line: $\psi_3 \exp(0.95i)$. The parameters are $p=1$, $k=0.9$, $u=2.8, 2.9, 3.1$, $\beta=0.49, 0.58, 0.74$, $r_i=1$, $i=1,3$.

$$\gamma = -\frac{k^2p^2}{2}\left(\frac{\mathrm{cn}^2(u)}{\mathrm{dn}^2(u)} - \frac{1}{k^2\mathrm{sn}^2(u)}\right),$$

$$\delta = \frac{\Theta_c'(u)}{\Theta_c(u)} + \frac{1}{2}\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)\mathrm{dn}(u)} - \frac{k^2\mathrm{sn}(u)\mathrm{cn}(u)}{\mathrm{dn}(u)}. \quad (27)$$

Here,

$$\Theta_c(u) = \theta_4\left(\frac{\pi u}{2(K-iK')}\right) = 1 + 2\sum (-1)^n q^{n^2}\cos\left(\frac{n\pi u}{K-iK'}\right), \quad (28)$$

with $q=\exp\{-\pi K'/(K-iK')\}$. A simple way to obtain (or check the correctness of) these results is by substituting $k \to 1/k$, $p \to kp$, $u \to ku$ in the results of Sec. III and using identities of elliptic functions like $\mathrm{dn}(ku,1/k)=\mathrm{cn}(u,k)$.

Figure 5 shows "a dark soliton + three bright solitons" lying on a cn background, which is obtained using Eq. (8) with $N=3$. The obtained solution has constant phases over z for fixed x values. In Fig. 5, the phase of ψ_0 [θ in Eq. (21)] is 0, while those of ψ_i, $i=1,3$, are −0.78, −0.83, −0.95, respectively. We thus draw $\psi_1 \exp(0.78i)$ for the dotted line of in Fig. 5(a), for example.

The shift of crests in this case is similarly calculated as in Appendix B. It is $\psi_0 \to \psi^{(0)} = kp\exp\{ip^2(2k^2-1)x\}\mathrm{cn}(pz)$ for $z\to\infty$ and $\psi_0 \to (-1)^N kp\exp\{ip^2(2k^2-1)x\}\mathrm{cn}(pz-2\Sigma_{i=1}^N u_i)$ for $z\to-\infty$, such that the shift of crest along the z axis is $2\Sigma u_i/p$.

As in the case of dn type, the shape of ψ_3 in Fig. 5 for $z\to\infty$ is sech type. It is described by Eq. (21) with (for simplicity, we take $u_3 \to u$)

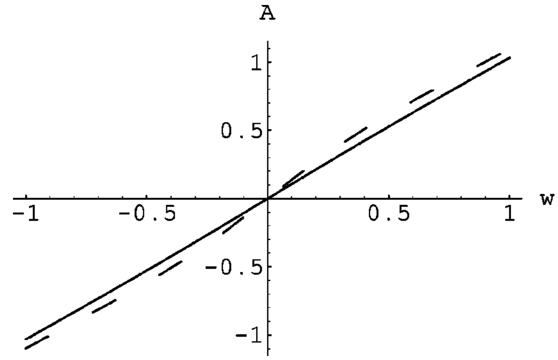

FIG. 6. Amplitude of the soliton on a cn background, amplitude vs width w; the solid line is for $k=0.5$, and the dashed line is for $k=0.9$ ($p=1$).

$$A = p\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)\mathrm{dn}(u)} \Bigg/ \sqrt{1 + k^2\frac{\mathrm{sn}^2(u)\mathrm{cn}^2(pz-u)}{\mathrm{dn}^2(u)}},$$

$$w = p\left(\mathrm{Re}\,\frac{\Theta_c'(u)}{\Theta_c(u)} + \frac{\mathrm{cn}(u)\mathrm{dn}(u)}{\mathrm{sn}(u)} - \frac{\pi K'}{2K}\frac{u}{K^2+K'^2}\right). \quad (29)$$

In the above expression, the last term of w originates from the following quasiperiodicity of the theta function:

$$\left|\frac{\Theta_c(pz-u+2K)}{\Theta_c(pz+2K)}\right| = \exp\left(-\pi\frac{uK'}{K^2+K'^2}\right)\left|\frac{\Theta_c(pz-u)}{\Theta_c(pz)}\right|. \quad (30)$$

A has a maximum value $p\,\mathrm{cn}(u)/[\mathrm{sn}(u)\mathrm{dn}(u)]$ and a minimum value $p\,\mathrm{cn}(u)/\mathrm{sn}(u)$. We thus define $\bar{A} \equiv p\,\mathrm{cn}(u)[1+\mathrm{dn}(u)]/[2\mathrm{sn}(u)\mathrm{dn}(u)]$. The amplitude A oscillates with the mean value $\bar{A}$ and periodicity $2K$. Especially as $k\to 0$, $\bar{A} \to p\cot u$. The width w is a monotonically decreasing function of u in $0 \leqslant u \leqslant 2K$, with $w=\infty$ at $u=0$, $w=0$ at $u=K$, and $w=-\infty$ at $u=2K$. In Fig. 6, we can see two plots, $\bar{A}$ vs w for a given k [Fig. 6(a); the solid line is for $k=0.5$, and the dotted line is for $k=0.9$] and $\bar{A}$ vs k for a given w [Fig. 6(b); the solid line is for $w=1$, and the dotted line is for $w=1.3$]. Figure 6 shows that the cn cnoidal wave with large k makes the amplitude of solitons large at a given width w. A plot of $\bar{A}$ vs k can be similarly drawn, which shows that the amplitude of a soliton on a cn background increases with k and drops sharply near $k=1$.

B. Type II

There is another type of MSC solutions on a cn-type background, which is obtained from the type-I case by substituting $u \to u + iK'$. In this case, the solution of the linear Equation (3) becomes

$$a = \exp\{ip^2(2k^2-1)x/2\}\exp(i\gamma x + \delta pz)\frac{\Theta_c(pz-u-iK')}{\Theta_c(pz)},$$

$$b = \exp\{-ip^2(2k^2-1)x\}\frac{\mathrm{dn}(pz-u)}{k\,\mathrm{cn}(u)\mathrm{sn}(pz-u)}a,$$

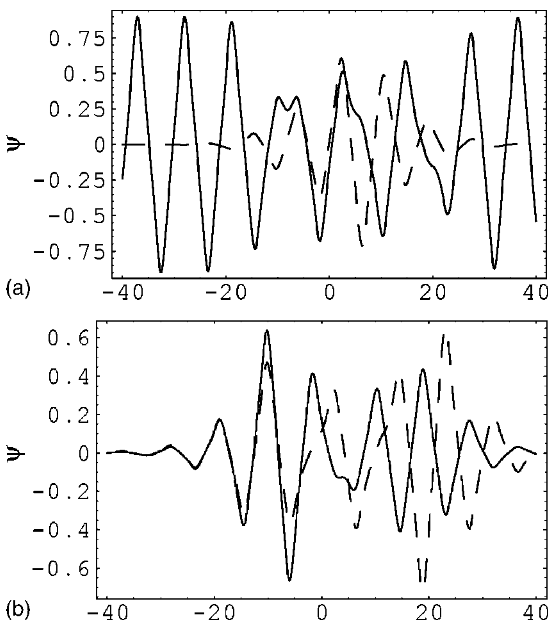

FIG. 7. Four-component soliton complexes constructed by adding three solitons on a cn background (type II). (a) Solid line: ψ_0. Dashed line: $\psi_1 \exp(-2.63i)$. (b) Solid line: $\psi_2 \exp(-2.84i)$. Dashed line: $\psi_3 \exp(0.20i)$. The parameters are $p=1$, $k=0.9$, $u=3.1, 2.9, 2.8$, $\beta=1.35, 1.73, 2.02$, $r_i=1$, $i=1,3$.

$$c = r \exp(\beta z/2 - i\beta^2 x/2). \tag{31}$$

The Bäcklund parameter β is

$$\beta = -p\frac{\mathrm{dn}(u)\mathrm{sn}(u)}{\mathrm{cn}(u)}, \tag{32}$$

and γ, δ in Eq. (31) are

$$\gamma = -\frac{p^2}{2}\left(\frac{\mathrm{dn}^2(u)}{\mathrm{cn}^2(u)} - k^2\,\mathrm{sn}^2(u)\right),$$

$$\delta = \frac{\Theta_c'(u+iK')}{\Theta_c(u+iK')} + \frac{1}{2}\frac{\mathrm{sn}(u)\mathrm{dn}(u)}{\mathrm{cn}(u)} - \frac{\mathrm{dn}(u)}{\mathrm{sn}(u)\mathrm{cn}(u)}. \tag{33}$$

Figure 7 shows "a dark soliton + three bright solitons" lying on a cn background, which is obtained using Eq. (8) with $N=3$. The obtained solutions have constant phases [θ in Eq. (21)] over z for a fixed x value. In Fig. 7, the phase of ψ_0 is 0, while those of ψ_i, $i=1,3$, are 2.63, 2.84, −0.20, respectively. We thus draw $\psi_1 \exp(-2.63i)$, $\psi_2 \exp(-2.84i)$, and $\psi_3 \exp(0.20i)$ in Fig. 7.

The shift of crests in the type-II case is calculated from $\psi_0 \to \psi^{(0)} = kp \exp\{ip^2(2k^2-1)x\}\mathrm{cn}(pz)$ for $z\to\infty$ and $\psi_0 \to kp \exp\{ip^2(2k^2-1)x\}\mathrm{cn}(pz - 2\Sigma_{i=1}^{N} u_i)$ for $z \to -\infty$, such that the shift is $2\Sigma u_i/p$.

The shape of ψ_3 in Fig. 7 is a sech type, described by Eq. (21) (with we take $u_3 \to u$)

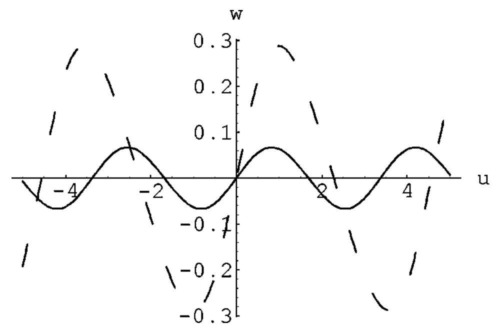

FIG. 8. Width w vs u for the soliton on a cn background (type II); the solid line is for $k=0.5$ and the dashed line is for $k=0.9$, $p=1$.

$$A = p\frac{\mathrm{sn}(u)\mathrm{dn}(u)}{\mathrm{cn}(u)} \Bigg/ \sqrt{1 + \frac{\mathrm{dn}^2(pz-u)}{k^2\mathrm{cn}^2(u)\mathrm{sn}^2(pz-u)}},$$

$$w = p\left(\mathrm{Re}\frac{\Theta_c'(u+iK')}{\Theta_c(u+iK')} - \frac{\mathrm{cn}(u)\mathrm{dn}(u)}{\mathrm{sn}(u)} - \frac{\pi K'}{2K}\frac{u+K}{K^2+K'^2}\right). \tag{34}$$

Contrary to the previous cases, the amplitude A becomes zero when $pz-u=2nK$. The extremum value of A is $\pm kp\,\mathrm{sn}u$ when $pz-u=(2n+1)K$. It gives the shape of ψ_3 in Fig. 7(b), which oscillates around zero value as $z\to\infty$. The width $|w|$ is an periodic function in u with a periodicity K, with maximum values at $u=(2n+1)K/2$, $n=$integer. In Fig. 8, we can see a plot of w vs u for a given k (solid line, $k=0.5$; dotted line, $k=0.9$). It shows that the width parameter w cannot be arbitrarily large (contrary to the cases of dn and cn type I), and the width of solitons in the type-II case has a minimum which depends on the modulus k.

V. MULTISOLITON COMPLEXES ON A sn-TYPE BACKGROUND

The cnoidal wave solution of the NLSE of the defocusing medium ($\sigma=-1$) is given by

$$\psi^{(0)} = \exp\{-ip^2(1+k^2)x\}ikp\,\mathrm{sn}(pz+K,k). \tag{35}$$

The linear equation (3) is integrated with the result

$$a = \exp\{-ip^2(1+k^2)x/2\}\exp(i\gamma x + \delta pz)\frac{\Theta_s(pz+K-u-iK')}{\Theta_s(pz)},$$

$$b = i\exp\{ip^2(1+k^2)x\}\frac{\mathrm{dn}(u)}{k\,\mathrm{sn}(pz-u)\mathrm{cn}(u)}a,$$

$$c = r\exp(\beta z/2 - i\beta^2 x/2). \tag{36}$$

The Bäcklund parameter β is given by

$$\beta = (k^2-1)p\frac{\mathrm{sn}(u)}{\mathrm{dn}(u)\mathrm{cn}(u)}, \tag{37}$$

and γ, δ in Eq. (36) are

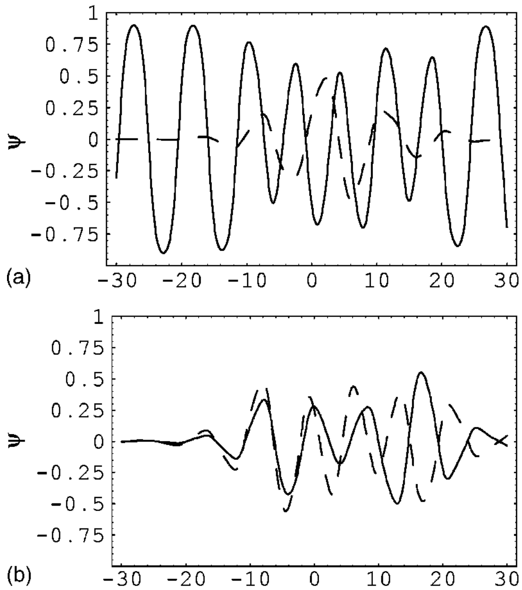

FIG. 9. Four-component MSC constructed by adding three solitons on a sn background. (a) Solid line: $-i\psi_0$. Dashed line: ψ_1. (b) Solid line: ψ_2. Dashed line: ψ_3. The parameters are $p=1$, $k=0.9$, $u=-2.8,-2.9,-3.1$, $\beta=-1.64,-1.29,-0.84$, $\rho_i=1$, $i=1,3$.

$$\gamma=\frac{(k^2-1)p^2}{2}\left(\frac{1}{\mathrm{cn}^2(u)}+k^2\frac{\mathrm{sn}^2(u)}{\mathrm{dn}^2(u)}\right),$$

$$\delta=\frac{\Theta_s'(u-K+iK')}{\Theta_s(u-K+iK')}+(1-k^2)\frac{\mathrm{sn}(u)}{2\,\mathrm{dn}(u)\mathrm{cn}(u)}-\frac{\mathrm{dn}(u)}{\mathrm{sn}(u)\mathrm{cn}(u)}. \tag{38}$$

Here,

$$\Theta_s(u)=\theta_4\left(\frac{i\pi u}{2K'}\right)=1+2\sum(-1)^n q^{n^2}\cos\left(\frac{in\pi u}{K'}\right), \tag{39}$$

with $q=-\exp(-\pi K/K')$. A simple way to obtain (or check the correctness of) these results is by substituting $k\to ik'/k$, $p\to ikp$, and $u\to iku$ in the results of Sec. III (MSC on a dn-type background) and using identities of elliptic functions like $\mathrm{dn}(ip,ik)=\mathrm{sn}(\sqrt{1+k^2}p+K(1/\sqrt{1+k^2}),1/\sqrt{1+k^2})$.

Figure 9 shows "a dark soliton + three bright solitons," which is obtained using Eq. (8) with $N=3$.

The shift of crests in this case is similarly calculated as in Appendix B, which is $\psi_0\to\psi^{(0)}=ikp\exp\{-ip^2(1+k^2)x\}\mathrm{sn}(pz+K)$ for $z\to-\infty$ and $\psi_0\to ikp\exp\{-ip^2(1+k^2)x\}\mathrm{sn}(pz+K-2\sum_{i=1}^N u_i)$ for $z\to\infty$.

The overall characteristics of solitons in this case are similar to the case of cn type II and are different from cases of dn and cn type I. The shape of ψ_3 in Fig. 9 is sech type, described by Eq. (21) with

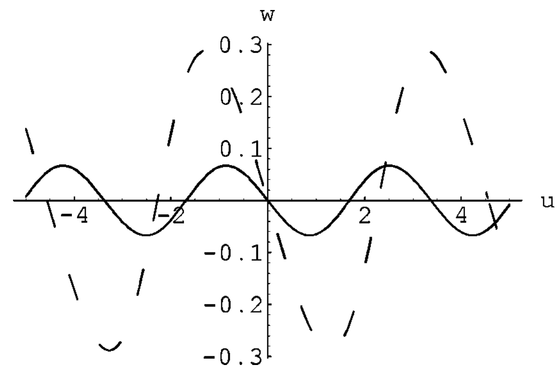

FIG. 10. Width parameter w of the soliton on a sn background: w vs u; the solid line is for $k=0.5$ and the dashed line is for $k=0.9$ ($p=1$).

$$A=k'^2p\frac{\mathrm{sn}(u)}{\mathrm{dn}(u)\mathrm{cn}(u)}\Bigg/\sqrt{-1+\frac{\mathrm{dn}^2(u)}{k^2\,\mathrm{cn}^2(u)\mathrm{sn}^2(pz-u)}},$$

$$w=p\left(\mathrm{Re}\frac{\Theta_s'(u-K+iK')}{\Theta_s(u-K+iK')}-\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)\mathrm{dn}(u)}+\frac{\pi}{2}\frac{K-u}{KK'}\right). \tag{40}$$

As in the case of cn type II, the width $|w|$ is a periodic function in u with a periodicity K, with maximum values at $u=(2n+1)K/2$, $n=$integer. The maximum value of $|A|$ is $pkk'\mathrm{sn}(u)/\mathrm{dn}(u)$. In Fig. 10, we can see a plot, w vs u for a given k (solid line, $k=0.5$; dotted line, $k=0.9$). It shows that the width parameter w cannot be arbitrarily large (contrary to the dn and cn type I), and a soliton in this case cannot be narrower than a given value.

VI. DISCUSSION

In this paper, we study the characteristics of MSC's lying on a cnoidal wave. It is connected with researches on the behaviors of solitons in periodic structures, which are intensely studied nowadays [35]. The analytic solutions of MSC's with cnoidal-wave backgrounds are obtained using the DT method. These solutions contain two important limits: (1) dn-type cnoidal waves become plane waves in the $k\to0$ limit, and (2) they become solitons in the $k\to1$ limit. Thus our solutions can be used to study the effect of cnoidal waves on MSC's in comparison to that of plane waves and/or coherent solitons. These solutions give important characteristics like amplitudes A and widths w of MSC's. A peculiar phenomenon of MSC solutions was the shift of crests of cnoidal waves.

The cnoidal waves used as starting solutions of the DT in the present paper are not the most general possible form of periodic solutions. In fact, there appear more complex periodic solutions expressed in terms of Weierstrass functions [36,37]. MSC's lying on these types of solutions would be interesting, because they can give more freedom in the control of MSC's.

The stability analysis of these solutions remains for future study. In fact, there already have appeared some numerical studies on this subject ($N=1$ case) [38,39]. There it was shown that the "a soliton+cnoidal wave" system is unstable or weakly stable for the focusing case, while it is stable for the defocusing case.

ACKNOWLEDGMENT

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APPENDIX A: PROOF OF SYM's SOLUTION

In this appendix, we show that a,b in Eq. (10) indeed satisfy the linear equation (3). Consider the following equation, which is the first ∂_z part of Eq. (3):

$$\partial_z a - \psi^{(0)} b + \beta a/2 = 0. \tag{A1}$$

By inserting a,b of Eq. (10) and β of Eq. (11) into Eq. (A1) and dividing it by pa, we get

$$\frac{\Theta'(u)}{\Theta(u)} + \frac{\Theta'(pz-u)}{\Theta(pz-u)} - \frac{\Theta'(pz)}{\Theta(pz)} - \frac{\mathrm{sn}(u)\mathrm{dn}(u)}{\mathrm{cn}(u)} + \frac{\mathrm{sn}(u)\mathrm{dn}(pz)\mathrm{dn}(pz-u)}{\mathrm{cn}(u)} = 0. \tag{A2}$$

Now, using the identities [32,40]

$$\int_0^u \mathrm{dn}^2(v)dv = \frac{\Theta'(u)}{\Theta(u)} + \frac{E}{K}u, \tag{A3}$$

the first three terms in Eq. (A2) become

$$\int_0^{pz-u} [\mathrm{dn}^2(v) - \mathrm{dn}^2(v+u)]dv. \tag{A4}$$

Finally, using the identity (it is a result of the addition theorem of Jacobi's elliptic functions)

$$\mathrm{dn}^2(a-b) - \mathrm{dn}^2(a) = -\frac{\mathrm{sn}(b)}{\mathrm{cn}(b)}\frac{d}{da}[\mathrm{dn}(a)\mathrm{dn}(a-b)], \tag{A5}$$

Eq. (A4) becomes

$$\frac{\mathrm{sn}(u)}{\mathrm{cn}(u)}[\mathrm{dn}(u) - \mathrm{dn}(pz-u)\mathrm{dn}(pz)]. \tag{A6}$$

Collecting all these results, it is now easy to see that Eq. (A2) becomes zero. The other remaining ∂_z parts of the linear equation (3) are similarly proved.

The first ∂_x part of Eq. (3) is

$$\partial_x a - i|\psi^{(0)}|^2 a - i\partial_z \psi^{(0)} b + i\beta\psi^{(0)} b - i\beta^2 a/2 = 0. \tag{A7}$$

By inserting a,b of Eq. (10) and β of Eq. (11) into Eq. (A7) and dividing it by $ip^2 a$, we get

$$\begin{aligned} 1 - \frac{1}{2}k^2 - \frac{\mathrm{dn}^2(u)}{2\,\mathrm{cn}^2(u)} + \frac{1}{2\,\mathrm{sn}^2 u} - \mathrm{dn}^2(pz) \\ - k^2\frac{\mathrm{sn}(u)}{\mathrm{cn}(u)}\mathrm{sn}(pz)\mathrm{cn}(pz)\mathrm{dn}(pz-u) \\ + \frac{\mathrm{dn}(u)}{\mathrm{cn}^2(u)}\mathrm{dn}(pz)\mathrm{dn}(pz-u) - \frac{\mathrm{dn}^2(u)}{2\mathrm{cn}^2(u)\mathrm{sn}^2(u)} \\ = (k^2-1)\frac{\mathrm{sn}^2(u)}{\mathrm{cn}^2(u)} - \mathrm{dn}^2(pz) + \frac{\mathrm{dn}(pz-u)}{\mathrm{cn}^2 u}[\mathrm{dn}(pz)\mathrm{dn}(u) \\ - k^2\,\mathrm{cn}(u)\mathrm{sn}(u)\mathrm{cn}(pz)\mathrm{sn}(pz)]. \end{aligned} \tag{A8}$$

Using the identities of elliptic functions including the addition theorem, the last two terms of Eq. (A8) become

$$\begin{aligned} -\mathrm{dn}^2(pz) + \frac{1}{\mathrm{cn}^2(u)}[\mathrm{dn}^2(u) - k^2\,\mathrm{cn}^2(u)\mathrm{sn}^2(pz)] \\ = -1 + \frac{\mathrm{dn}^2(u)}{\mathrm{cn}^2(u)}. \end{aligned} \tag{A9}$$

It is now easy to see that Eq. (A8) becomes zero, which proves Eq. (A7). The other ∂_x parts of the linear equation (3) are similarly proved.

APPENDIX B: ASYMPTOTIC FORM OF ψ_0

For the parameters of Fig. 1, $\delta p=0.60$, $\beta_1/2=0.94$ and Eq. (10) shows $|c| \gg |a|,|b|$ for $z\to\infty$. Then, $D=LP_1 \approx |c|^2$ and $Q_0/D \approx 2i\beta ab^*/|c|^2 \to 0$ such that $\psi_0 \to \psi^{(0)} = p\exp\{ip^2(2-k^2)x\}\mathrm{dn}(pz)$. On the other hand, $|c| \ll |a|,|b|$ for $z\to -\infty$ such that $D=LP_1 \approx |a_1|^2+|b_1|^2$ and

$$\begin{aligned} iQ_0/D &\approx -2\beta\left(\frac{b}{a} + \frac{a^*}{b^*}\right)^{-1} \\ &= -2p\exp\{ip^2(2-k^2)x\}\frac{\mathrm{dn}(u)\mathrm{dn}(pz-u)}{\mathrm{cn}^2(u) + \mathrm{sn}^2(u)\mathrm{dn}^2(pz-u)}. \end{aligned} \tag{B1}$$

Then,

$$\begin{aligned} \psi(z\to-\infty) &= p\exp\{ip^2(2-k^2)x\}\left(\mathrm{dn}(pz) - 2\frac{\mathrm{dn}(u)\mathrm{dn}(pz-u)}{\mathrm{cn}^2(u) + \mathrm{sn}^2(u)\mathrm{dn}^2(pz-u)}\right) \\ &= p\exp\{ip^2(2-k^2)x\}\left[\frac{\mathrm{dn}(pz-u)\mathrm{dn}(u) - k^2\mathrm{sn}(pz-u)\mathrm{sn}(u)\mathrm{cn}(pz-u)\mathrm{cn}(u)}{1-k^2\mathrm{sn}^2(pz-u)\mathrm{sn}^2(u)} - 2\frac{\mathrm{dn}(u)\mathrm{dn}(pz-u)}{1-k^2\mathrm{sn}^2(pz-u)\mathrm{sn}^2(u)}\right] \\ &= -p\exp\{ip^2(2-k^2)x\}\mathrm{dn}(pz-2u). \end{aligned} \tag{B2}$$

In the last part of the above derivation, we use the addition theorem of elliptic functions.

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